Engineering Notes

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Analysis of a Partially Cracked Panel

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Nomenclature

$2b$ = panel width
E = modulus of elasticity
e_{ij} = strain tensor
G = shear modulus of elasticity
l = length of a panel
$2L$ = crack length
u, v = x and y components of displacement
W_s = band width
x, y = Cartesian coordinates
σ_{ij} = stress tensor
θ = first strain invariant
δ_{ij} = Kronecker delta
σ_a = applied stress

MODERN aircraft structural design takes into account the fail safe consideration. In the giant C-5A structural design, an intermediate titanium band is bonded on the skin between every two adjacent frames for reduction of peak stress level on one hand and for the fail safe design on the other. A crack may be allowed to occur between two adjacent straps and across a frame without causing catastrophic failure of the whole structural system. The proper design of such structural panel adequate for fail safe consideration depends on the knowledge of the stress redistribution.

Each panel of $l \times 2b$ size is attached to stiffeners along its edges $x = 0$, l, and $y = \pm b$. The stiffeners are considered to be rigid in the longitudinal direction and flexible laterally. A thin strap of W_s width is bonded on each panel between two adjacent stiffeners as a crack stopper, and a crack is considered to occur along $y = 0$ and stopped at the edge of straps of two adjacent panels similarly loaded along $y = \pm b$. The problem is treated as a plane elasticity problem and the effects caused by the variation of panel thickness due to bands are considered negligible. Solutions satisfying exactly the governing differential equations and all but one boundary condition are obtained. The last boundary condition which is one of the two conditions along the crack line is made to be satisfied by collocation approximation. The bandwidth is then determined according to an engineering approach by considering that the panel material within the bandwidth becomes fully plastic. Rigorous elastoplastic analysis is not considered.

A numerical example based on typical C-5A panel material and geometry is included for illustrative purposes. Curves relating the applied stress and minimum required bandwidth are presented.

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Elastic Stress Analysis

Basic equations for plane elasticity problems may be found in books on elasticity such as Ref. 1. The stress displacement relations and equilibrium equations governing displacement components according to the Hooke's law are presented below in tensor notation:

$$\sigma_{ij} = \frac{E}{1-\nu^2}\left[\nu u_{k,k}\delta_{ij} + \frac{1}{2}(1-\nu)(u_{i,j}+u_{j,i})\right] \quad (1)$$

$$\tfrac{1}{2}(1-\nu)u_{i,jj} + \tfrac{1}{2}u_{j,ij} = 0 \quad (2)$$

where σ_{ij} is the stress tensor, u_i are displacement components, E is the modulus of elasticity, ν is the Poisson's ratio, and δ_{ij} is the Kronecker delta. In expanded form, Eq. (2) with $u_1 = u$, $u_2 = v$, $x_1 = x$ and $x_2 = y$ may be written as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} = 0 \quad (3a)$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} = 0 \quad (3b)$$

Because of symmetry in geometry and loading about $x = 0$ and $y = 0$, only half of a panel ($0 \le x \le l$, $0 \le y \le b$) as shown in Fig. 1 is needed for analysis. According to the type of edge stiffeners and loading condition described earlier, the following boundary conditions are considered:

$$u(0, y) = 0 \quad (4)$$

$$u(l, y) \neq 0 \quad (5)$$

$$v(0, y) = 0 \quad (6)$$

$$v(l, y) = 0 \quad (7)$$

$$u(x, b) = 0 \quad (8)$$

$$\sigma_{yy}(x, b) = \sigma_a(x) \quad \text{or} \quad \frac{\partial v}{\partial y} = \frac{1-\nu^2}{E}\sigma_a(x) \quad \text{for} \quad y = 0 \quad (9)$$

$$\tau_{xy}(x, 0) = 0 \quad \text{or} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \text{for} \quad y = 0 \quad (10)$$

$$v(x, 0) = 0 \quad \text{for} \quad L \le x \le l \quad (11)$$

$$\sigma_{yy}(x, 0) = 0 \quad \text{for} \quad 0 < x < L \quad (12)$$

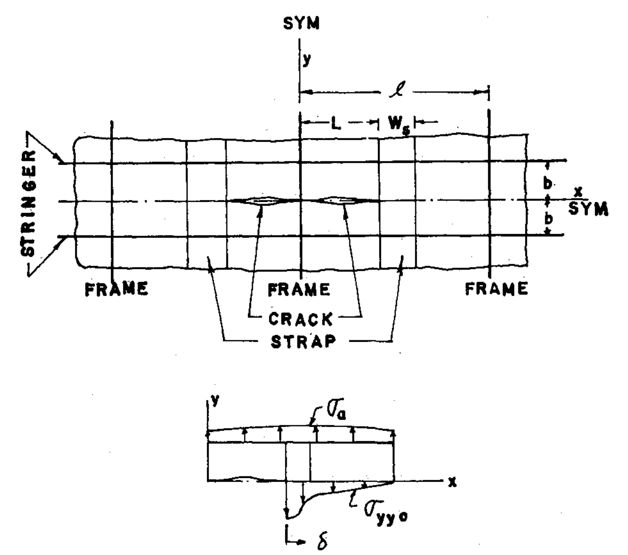

Fig. 1 Geometry and coordinates.

According to the boundary conditions given by Eqs. (4)–(7), the general solutions of the differential Eqs. (3a) and (3b) are taken to be

$$u = \sum_{m=1}^{\infty} U_m(y)(\cos\alpha_m x - 1) \tag{13}$$

$$v = \sum_{m=1}^{\infty} V_m(y) \sin\alpha_m x \tag{14}$$

where $\alpha_m = m\pi/l$. Substituting Eqs. (13) and (14) into Eqs. (3a) and (3b) and subsequently solving the resulting differential equations, one obtains, for each m, the following general solutions:

$$V_m = (A_m + C_m\alpha_m y)\cosh\alpha_m y + (B_m + D_m\alpha_m y)\sinh\alpha_m y \tag{15}$$

$$U_m = \left(A_m + C_m\alpha_m y + \frac{3-\nu}{1+\nu} D_m\right)\sinh\alpha_m y + \left(B_m + \frac{3-\nu}{1+\nu} C_m + D_m\alpha_m y\right)\cosh\alpha_m y \tag{16}$$

Detailed steps may be referred to in Ref. 2.

By satisfying the boundary conditions (8–10), one obtains the following results:

$$D_m = -(1+\nu)/2\, A_m \tag{17}$$

$$C_m = \beta_m A_m + \beta_m{}^* B_m \tag{18}$$

$$\gamma_m{}^* B_m = a_m/E\alpha_m - \gamma_m A_m \tag{19}$$

where β_m, $\beta_m{}^*$, Δ_m, γ_m, and $\gamma_m{}^*$ are constants, and

$$a_m = \frac{2}{l}(1-\nu^2)\int_0^l \sigma_a(x)\sin\alpha_m x\, dx$$

Now all the unknown coefficients can be expressed in terms of one set of Fourier coefficients A_m. These constants may be determined by using the boundary conditions (11) and (12), and these conditions require

$$\sum_{m=1}^{\infty}\left(\delta_m \bar{A}_m + \frac{a_m}{\gamma_m{}^*} g_m\right)\sin\alpha_m x = 0 \qquad \text{for} \qquad 0 < x < L \tag{20}$$

$$\sum_{m=1}^{\infty} \bar{A}_m \sin\alpha_m x = 0 \qquad \text{for} \qquad L \le x \le l \tag{21}$$

to be satisfied where

$$\bar{A}_m = EA_m;\ \delta_m = \alpha_m\left[\beta_m \frac{(1-\nu)^2}{1+\nu} - \frac{\gamma_m}{\gamma_m{}^*} g_m\right]$$

$$g_m = 1 + \beta_m{}^* - \nu\left(1 + \frac{3-\nu}{1+\nu}\beta_m{}^*\right)$$

It is difficult to satisfy Eqs. (20) and (21) for all values of x, the collocation method requiring the satisfaction of Eqs. (20) and (21) at a number of points will be used. As a result, the number of A_m equals the number of points considered in the collocation method will be used, and Eqs. (20) and (21) become

$$\sum_{m=1}^{N} (\delta_m \sin\alpha_m x_j)\bar{A}_m = -\sum_{m=1}^{N} \frac{a_m}{\gamma_m{}^*} g_m \sin\alpha_m x_j \tag{22}$$

$$\sum_{m=1}^{N} \bar{A}_m \sin\alpha_m x_k = 0 \tag{23}$$

for $j = 1, 2, 3, \ldots, J$, and $k = 1, 2, 3, \ldots k$ where N is equal to the total number of collocation points considered, J represents the number of points considered in the region $0 < x < L$, and K represents the number of points considered in the region $L \le x \le l$. Equations (22) and (23) represent a total of N simultaneous algebraic equations from which N number of A_m can be determined. Some discussions on the collocation method may be found in Ref. 2. If no crack occurs in the panel, $L = 0$, and $A_m = 0$ according to Eq. (21). After having determined A_m, the displacement components can be computed according to Eqs. (13) and (14) in conjunction with Eqs. (15–19), and the stresses can be computed according to Eq. (1).

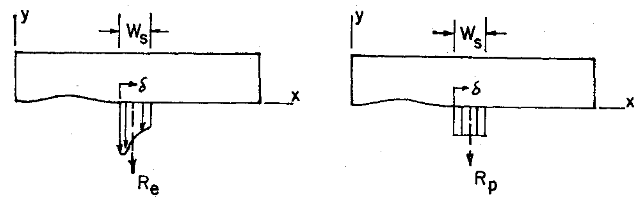

Fig. 2 Approximation in stress distribution of strap location.

Determination of Bandwidth or Maximum Admissible Applied Stress

It is assumed that for a given bandwidth W_s the magnitude of the resultant force R_e of the stresses according to the linear elasticity analysis in the region of the bandwidth equals the magnitude of the resultant force R_p in the same region when the material becomes plastic in this region. This is illustrated in Fig. (2). According to this assumption $R_e = R_p$, where

$$R_e = \int_L^{L+W_s} \sigma_{yy0}\, dx;\quad R_p = \sigma_p W_s;\quad \sigma_{yy0} = \sigma_{yy|y=0}$$

and where σ_p represents the yield stress of the material. If $\bar{\sigma}_a$ represents the amplitude of the applied stress σ_a along $y = b$, and if $\bar{\sigma}_{yy0}$ represents σ_{yy0} corresponding to $\bar{\sigma}_a =$ unity, one obtains

$$\bar{\sigma}_a \int_L^{L+W_s} \bar{\sigma}_{yy0}\, dx = \sigma_p W_s \tag{24}$$

Equation (24) gives the relationship between the maximum admissible applied stress $\bar{\sigma}_a$ and the bandwidth W_s.

Numerical Examples

For illustrative purposes, numerical results for uniformly applied stress σ_a with material and geometry similar to C-5A fuselage panels are obtained according to the data $l = 20$ in., $b = 3.9$ in., $\nu = 0.3$, $W_s = 0.5$, 1.5, 2.5, ... 11.5 in. The normal stress variation along $y = 0$ computed according to 20, 40 and 80 collocation points for $W_s = 4.5$ in. are plotted in Fig. 3. The discrepancies occur near the very vicinity of the crack tip and diminish sharply. Since the region near the crack tip is always in the plastic state, the yield stress will actually be considered. The bandwidth calculated according to the analysis presented earlier will be the same for 20, 40, and 80 equal spaced collocation point approximations. Forty point approximation is used for all other results pre-

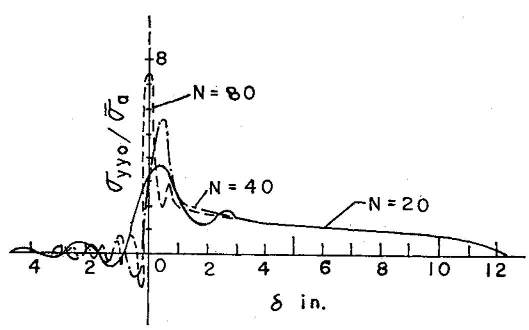

Fig. 3 Stress variation along the crack line.

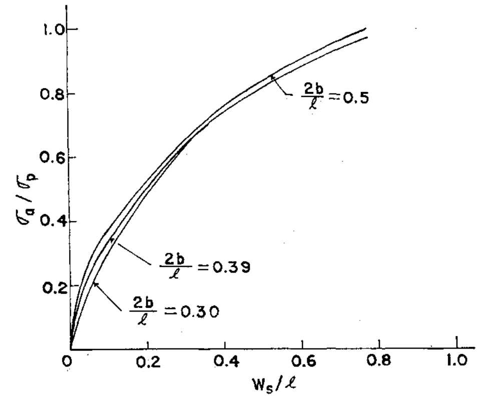

Fig. 4 Maximum admissible applied stress vs strap width.

sented in Fig. 4. Determination of more realistic applied stress distribution for fuselage panels may be found in the analysis presented in Ref. 4.

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Higher Vibration Modes by Matrix Iteration

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Introduction

THE method of matrix iteration remains a useful approach to determining normal modes of vibration for elastic structures. It is straightforward to use, with or without the aid of a digital computer, and it converges rapidly if the natural frequencies are well separated. It possesses the disadvantage that numerical errors in lower vibration modes are propagated into the calculations for higher modes. This difficulty can be overcome, however, by employing a hybrid method which alternately searches for zeros in the characteristic determinant in the neighborhood of frequencies found by iteration.

The usual method of finding higher modes by matrix iteration is sweeping, which is described in textbooks on structural dynamics.[1–3] In the sweeping technique a matrix must be generated which renders any trial matrix for, say, the kth vibration mode orthogonal to the first $k-1$ modes. Thus, orthogonality of modes is assured (to within the numerical accuracy implied) and the dynamic equations are solved by iteration.

Another method for finding higher modes can be devised which satisfies both the orthogonality relations and the dynamic equations simultaneously by iteration. It stems from the result presented without proof on pages 168–169 of Ref. 1; it was communicated to these authors by M. J. Turner of the Boeing Airplane Company. This approach is the subject of this Note, and it will be referred to as "Turner's method."

Derivation of Turner's Method

Consider a dynamic system characterized by n-degrees of freedom. Assume for illustrative purposes that the first vibration mode $\boldsymbol{\varphi}^{(1)}{}_{n\times 1}$ and its natural frequency ω_1 have been found by some means and that it is desired to find the second mode $\boldsymbol{\varphi}^{(2)}$ and its corresponding frequency ω_2 by matrix iteration. The second mode must satisfy the orthogonality relation

$$(\boldsymbol{\varphi}^{(1)T}{}_{1\times n})(M_{n\times n})\boldsymbol{\varphi}^{(2)}{}_{n\times 1} = 0 \tag{1}$$

and the dynamic equation

$$(\mathbf{D}_{n\times n})\boldsymbol{\varphi}^{(2)}{}_{n\times 1} = [1/(\omega_2)^2]\boldsymbol{\varphi}^{(2)}{}_{n\times 1} \tag{2}$$

$\mathbf{M}$ is the system's mass matrix and $\mathbf{D} = \mathbf{CM}$. $\mathbf{C}$ is the flexibility matrix for the structure.

A modified iteration problem can be defined of the following form:

$$[\mathbf{D} - (\mathbf{B}_{n\times 1})(\boldsymbol{\varphi}^{(1)})^T\mathbf{M}]\mathbf{A}_{n\times 1} = (1/\omega^2)\mathbf{A}_{n\times 1} \tag{3}$$

$\mathbf{A}$ is a matrix of modal amplitudes and $\mathbf{B}$ is a matrix whose form is as yet unspecified. Notice that if we set $\mathbf{A} = \boldsymbol{\varphi}^{(2)}$ and $\omega = \omega_2$ Equation (3) will be satisfied for any nonzero $\mathbf{B}$ matrix. We will choose a $\mathbf{B}$-matrix that will insure that the iteration of this equation will converge to $\boldsymbol{\varphi}^{(2)}$ and ω_2.

Any trial vector $\mathbf{A}$ can be expressed as a linear combination of the n true modes of the form

$$\mathbf{A} = \sum_{k=1}^{n} a_k\boldsymbol{\varphi}^{(k)} = \underset{n\times n}{\begin{bmatrix} \boldsymbol{\varphi}^{(1)} \boldsymbol{\varphi}^{(2)} \cdots \boldsymbol{\varphi}^{(n)} \\ \vdots \quad \vdots \quad\quad \vdots \end{bmatrix}} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = (\boldsymbol{\Phi}_{n\times n})(\mathbf{a}_{n\times 1}) \tag{4}$$

$\boldsymbol{\varphi}$ is the modal matrix composed of columns of vibration modes and $\mathbf{a}$ is the matrix of modal amplitudes. $\boldsymbol{\varphi}$ satisfies the equation

$$\mathbf{D\Phi} = \boldsymbol{\Phi}\begin{bmatrix} 1/(\omega_1)^2 & 0 & 0 \\ 0 & 1/(\omega_2)^2 & \\ & & \ddots \\ & 0 & 1/(\omega_n)^2 \end{bmatrix} \tag{5}$$

Also

$$\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & \\ & & \ddots \\ & 0 & M_n \end{bmatrix} \tag{6}$$

where the M_k are generalized masses defined as

$$M_k = (\boldsymbol{\varphi}^{(k)})^T\mathbf{M}\boldsymbol{\varphi}^{(k)} \tag{7}$$

If Eq. (4) is substituted into the left-hand side of Eq. (3), we obtain

$$\mathbf{D\Phi}a - \mathbf{B}(\boldsymbol{\varphi}^{(1)})^T\mathbf{M\Phi a} \tag{8}$$

$$= \boldsymbol{\Phi}\begin{bmatrix} 1/(\omega_1)^2 & & \\ & 1/(\omega_2)^2 & \\ & & \ddots \\ & & 1/(\omega_n)^2 \end{bmatrix}\mathbf{a} - \mathbf{B}[\underbrace{M_1 0 0 \ldots 0}_{n\text{-terms}}]\mathbf{a}$$

$$= \left(\frac{1}{(\omega_1)^2}\boldsymbol{\varphi}^{(1)} - M_1\mathbf{B}\right)a_1 + \sum_{k=2}^{n}\frac{1}{(\omega_k)^2}\boldsymbol{\varphi}^{(k)}a_k$$

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